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Beckmann's Bertrand–Edgeworth Duopoly Model: New Pure Strategy Equilibria

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Abstract—In this paper, the concept of weak active equilibrium, which was introduced in the eighties of the twentieth century by Smol'yakov, is used for the first time for a study of Bertrand–Edgeworth oligopoly (that is, competition among firms when firms' strategies are prices and firms' production capacities are limited). All symmetric weak active equilibria are found for the basic model of Bertrand–Edgeworth duopoly, which was analysed in the sixties of the twentieth century by Beckmann. However, this model is highly simplified. Also, the question of existence of nonsymmetric weak active equilibria is studied.

Keywords: Bertrand–Edgeworth duopoly, noncooperative game, weak extremal profile of actions, weak active equilibrium

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1. INTRODUCTION

For its simplicity, the Beckmann's model [1, 2] is far enough from reality. However, this model has served as a starting point for many subsequent studies (e.g., [3–15]). In the classical setting for Bertrand oligopoly, it is assumed that, after the announcement of prices, firms can produce as much as necessary to meet demands. In this setting, firms' strategies are prices and the firm's goal is to maximize profits. The assumption of bounded capacities of firms (in this case the Bertrand oligopoly is called the Bertrand–Edgeworth oligopoly) significantly changes the answer to the question what strategies should be chosen by firms in this game. For the model in question, Beckmann proved that Nash equilibria in pure strategies exist for some firms' capacities and do not exist for other firms' capacities. For the latter case, Beckmann found Nash equilibria in mixed strategies. This led to the fact that considerable attention is given to Nash equilibria in mixed strategies in many subsequent studies of Bertrand–Edgeworth oligopolies. Despite the fact that, in studies of oligopolies, Nash equilibria in pure strategies are more important than Nash equilibria in mixed strategies for practical purposes.

The present paper shows that equilibria in pure strategies exist for all firms' capacities in the Beckmann's model if weak active equilibria [16, 17] are used instead of Nash equilibria. (The Nash equilibrium is a special case of the weak active equilibrium.) So, there is no need for equilibria in mixed strategies when studying this duopoly model. In this paper, it is shown which symmetric profiles of actions are weak active equilibria and which ones are not. Also, the question of existence of nonsymmetric weak active equilibria is investigated. In more detail, Cournot oligopoly and Bertrand oligopoly are described, for instance, in [18].

Suppose that each of the indices i and j takes the values 1 and 2, $i \neq j$. The player i can choose an action x_i from a set D_i. By $f_i(x_i, x_j)$ denote the payoff of the player i for the profile (x_i, x_j) of actions.

Definition 1. The profile $(\bar{x}_i, \bar{x}_j)$ of actions is called weak extremal for the player i if for any action $x_i \in D_i$ there exists an action $x_j \in D_j$ such that

$$f_i(x_i, x_j) \leqslant f_i(\bar{x}_i, \bar{x}_j).$$

Definition 2. The profile $(\bar{x}_i, \bar{x}_j)$ of actions is called a weak active equilibrium if this profile is weak extremal for any player.

The concepts of the weak extremal profile and the weak active equilibrium were introduced in the eighties of the twentieth century by Smol'yakov.

Definition 3. The profile (x_i^*, x_j^*) of actions is called a Nash equilibrium if

$$f_i(x_i^*, x_j^*) = \max_{x_i \in D_i} f_i(x_i, x_j^*)$$

for any i.

It is not difficult to see that any Nash equilibrium is a weak active equilibrium. In general, the converse is not true.

In Section 2, Bertrand–Edgeworth duopoly is described. In Section 3, a necessary and sufficient condition for a profile to be weak extremal is given for small production capacities. The same condition is given for big production capacities in Section 4. In Section 5, weak active equilibria are presented and practical recommendations are discussed.

2. BERTRAND–EDGEWORTH DUOPOLY

Two firms that produce a homogeneous good are under consideration. Production costs are assumed to be zero. Production capacities of the firms are equal. A quantity of goods that a firm can produce is denoted by c, $0 < c < 1$. The firms simultaneously announce prices $x_1 \in D_1$ and $x_2 \in D_2$ at which the product will be sold by the firm, $D_1 = D_2 = [0, 1]$. The demand function is assumed to be linear:

$$q = 1 - x. \tag{1}$$

This means that if the firms announced the same price x, then the quantity of sold goods is determined by formula (1). Then the income of each firm is equal to $0.5x(1-x)$. (It is assumed that the firms share the market equally in this case.) If the firms announced different prices, then the firm that announced a lower price sells the good first; by x denote this price. Suppose that $x > 1 - c$. Then it follows from (1) that $q < c$. Thus, the quantity of goods that is sold by the firm is equal to $1 - x$. In this case, the other firm sells nothing and the income of the other firm is zero. Suppose that $x \leqslant 1 - c$. Then it follows from (1) that $q \geqslant c$. The quantity of goods that is sold by the firm is equal to c due to limited production capacities. By y denote the price announced by the other firm. The residual demand function for the other firm has the form

$$\left(1 - \frac{c}{1-x}\right)(1-y). \tag{2}$$

This means that the quantity of goods that can be sold by the other firm is determined by formula (2). Economic meaning of the residual demand function is discussed, for instance, in [5]. Also, [5] compares (2) and another form of the residual demand function. The goal of the firm is maximize its income which is calculated as the product of the price and the quantity of the good that is sold. It can be assumed that the quantity of the good that is produced by each firm is equal to c since the production costs are zero.

Thus, the income of the firm 1 is calculated by the following formula

$$f(x_1, x_2) = \begin{cases} cx_1 & \text{for} \quad x_1 < x_2,\ x_1 \leqslant 1 - c, \\ x_1(1 - x_1) & \text{for} \quad x_1 < x_2,\ x_1 > 1 - c, \\ cx_1 & \text{for} \quad x_1 = x_2,\ x_2 \leqslant 1 - 2c, \\ 0.5x_1(1 - x_1) & \text{for} \quad x_1 = x_2,\ x_2 > 1 - 2c, \\ x_1 \min\left(c, \left(1 - \dfrac{c}{1 - x_2}\right)(1 - x_1)\right) & \text{for} \quad x_1 > x_2,\ x_2 \leqslant 1 - c, \\ 0 & \text{for} \quad x_1 > x_2,\ x_2 > 1 - c. \end{cases} \tag{3}$$

The income of the firm 2 is equal to $f(x_2, x_1)$. Zero prices are allowed to be announced. But it turns out that such case is of little interest for this model in contrast to classical Bertrand oligopoly.

It is not difficult to show that the profile $(1 - 2c, 1 - 2c)$ is a Nash equilibrium if $0 < c \leqslant 0.25$ ([1, 2]). In accordance with (1), this means that demand is satisfied as much as firms' production capacities permit. Also, it is not difficult to show that the profile $(1 - 2c, 1 - 2c)$ is not a Nash equilibrium for $0.25 < c < 0.5$ ([1, 2]).

3. WEAK EXTREMAL PROFILES FOR SMALL PRODUCTION CAPACITIES

Denote

$$\mu = (1 - 2c)c.$$

Lemma 1. *Assume that* $0 < c \leqslant \frac{1}{3}$. *The profile* $(\bar{x}_1, \bar{x}_2)$ *is weak extremal for the firm 1 if and only if* $f(\bar{x}_1, \bar{x}_2) \geqslant \mu$.

Proof. Suppose that $f(\bar{x}_1, \bar{x}_2) \geqslant \mu$. Assume that $0 \leqslant x_1 \leqslant 1 - 2c$. Then for any x_2 we have

$$f(x_1, x_2) = cx_1 \leqslant (1 - 2c)c \leqslant f(\bar{x}_1, \bar{x}_2).$$

Assume that $1 - 2c < x_1 \leqslant 1 - c$. If we consider the limit on the left we find

$$\lim_{x_2 \to x_1 - 0} \left(1 - \frac{c}{1 - x_2}\right) x_1(1 - x_1) = \left(1 - \frac{c}{1 - x_1}\right) x_1(1 - x_1) = (1 - c - x_1)x_1.$$

Consequently,

$$\lim_{x_2 \to x_1 - 0} f(x_1, x_2) = (1 - c - x_1)x_1$$

because of the inequality $1 - c - x_1 < c$. Since the function $\left(1 - \frac{c}{1 - x_2}\right)$ is continuous and monotone decreasing for $x_2 < x_1$, for any $\varepsilon > 0$ there exists $x_2 < x_1$ such that

$$f(x_1, x_2) \leqslant (1 - c - x_1)x_1 + \varepsilon.$$

Value of the function $(1 - c - x_1)x_1$ is equal to $(1 - 2c)c$ for $x_1 = 1 - 2c$. The function $(1 - c - x_1)x_1$ is monotone decreasing for $x_1 > 1 - 2c$ because of the condition $c \leqslant \frac{1}{3}$. Therefore $(1 - c - x_1)x_1 < \mu$. Consequently, there exists ε such that $f(x_1, x_2) \leqslant \mu$ for some $x_2 < x_1$.

Assume that $1 - c < x_1 \leqslant 1$. Then $f(x_1, 1 - c) = 0 < \mu$.

Suppose that $f(\bar{x}_1, \bar{x}_2) < \mu$. The profile $(\bar{x}_1, \bar{x}_2)$ is not weak extremal for the firm 1 since $f(1 - 2c, x_2) = \mu$ for any x_2.

This completes the proof of Lemma 1.

4. WEAK EXTREMAL PROFILES FOR BIG PRODUCTION CAPACITIES

Denote

$$\nu = \left(\frac{1-c}{2}\right)^2.$$

Lemma 2. *Assume that* $\frac{1}{3} < c < 1$. *The profile* $(\bar{x}_1, \bar{x}_2)$ *is weak extremal for the firm 1 if and only if* $f(\bar{x}_1, \bar{x}_2) > \nu$.

Proof. Suppose that $f(\bar{x}_1, \bar{x}_2) > \nu$. Assume that $0 \leqslant x_1 \leqslant \max(0, 1-2c)$. Then

$$f(x_1, x_2) = cx_1 \leqslant c \max(0, 1-2c) \leqslant \left(\frac{1-c}{2}\right)^2 < f(\bar{x}_1, \bar{x}_2)$$

for $x_2 > x_1$.

Assume that $\max(0, 1-2c) < x_1 \leqslant 1-c$. Denote $\varepsilon = f(\bar{x}_1, \bar{x}_2) - \nu$. We have

$$\lim_{x_2 \to x_1 - 0} \left(1 - \frac{c}{1-x_2}\right) x_1(1-x_1) = (1-c-x_1)x_1.$$

Consequently,

$$\lim_{x_2 \to x_1 - 0} f(x_1, x_2) = (1-c-x_1)x_1$$

since $1-c-x_1 < c$. The function $(1-c-x_1)x_1$ reaches the maximum at the point $x_1 = \frac{1-c}{2}$ (an interior point of the segment $[\max(0, 1-2c), 1-c]$); the maximum is $\left(\frac{1-c}{2}\right)^2$. Thus, for any $x_1 \in (\max(0, 1-2c), 1-c]$ there exists $x_2 < x_1$ such that

$$f(x_1, x_2) < \left(\frac{1-c}{2}\right)^2 + \frac{\varepsilon}{2} < f(\bar{x}_1, \bar{x}_2).$$

Assume that $1-c < x_1 \leqslant 1$. Then $f(x_1, 1-c) = 0 \leqslant \nu$.

We now prove that the profile $(\bar{x}_1, \bar{x}_2)$ is not weak extremal for the firm 1 if $f(\bar{x}_1, \bar{x}_2) \leqslant \nu$. It is sufficient to show that $f\left(\frac{1-c}{2}, x_2\right) > \nu$ for any x_2. Note that $\max(0, 1-2c) < x_1 \leqslant 1-c$ for $x_1 = \frac{1-c}{2}$ since $c > \frac{1}{3}$.

Assume that $x_2 < x_1$. The function $f(x_1, x_2)$ considered as a function of the argument x_2 is monotone nonincreasing for $x_2 \in [0, x_1)$. As was established above in proving this lemma, the function $f(x_1, x_2)$ coincides with the function $\left(1 - \frac{c}{1-x_2}\right) x_1(1-x_1)$ near the point x_1. Therefore the function is monotone decreasing. Thus,

$$f(x_1, x_2) > (1-c-x_1)x_1 = \left(\frac{1-c}{2}\right)^2.$$

Assume that $x_2 = x_1$. Then

$$f(x_1, x_2) = 0.5x_1(1-x_1) = \frac{1-c}{2} \times \frac{1+c}{4} > \left(\frac{1-c}{2}\right)^2,$$

since $c > \frac{1}{3}$. Assume that $x_2 > x_1$. Then

$$f(x_1, x_2) = c\,\frac{1-c}{2} > \left(\frac{1-c}{2}\right)^2,$$

since $c > \frac{1}{3}$.

This completes the proof of Lemma 2.

5. WEAK ACTIVE EQUILIBRIA

By (3) it follows that if $0 < c \leqslant \frac{1}{4}$, then $f(x,x) = \mu$ for $x = 1-2c$ and $f(x,x) < \mu$ for $x \neq 1-2c$. Thus, by Lemma 1, $(1-2c, 1-2c)$ is the unique profile (x,x), which is a weak active equilibrium, for $0 < c \leqslant \frac{1}{4}$. Also, this profile is a Nash equilibrium.

Proposition 1. *Assume that* $\frac{1}{4} < c \leqslant \frac{1}{3}$. *A profile* (x,x) *is a weak active equilibrium if and only if* $1-2c \leqslant x \leqslant 2c$.

Proof. The proof is reduced to use of Lemma 1 and check of condition $0.5x(1-x) \geqslant \mu$, which is equivalent to $1-2c \leqslant x \leqslant 2c$.

This completes the proof of Proposition 1.

The symmetric pure strategy equilibrium $(1-2c, 1-2c)$ was found in [15], Proposition 4.3, for $\frac{1}{4} < c \leqslant \frac{1}{3}$. Also, it is established in [15] that there are no other equilibria in secure strategies for this game. However, by (3) it follows that the income of the firm is higher for the profile $(0.5, 0.5)$, which is a weak active equilibrium, than for the profile $(1-2c, 1-2c)$ for $\frac{1}{4} < c \leqslant \frac{1}{3}$.

Proposition 2. *Assume that* $\frac{1}{3} < c < 1$. *A profile* (x,x) *is a weak active equilibrium if and only if*

$$\frac{1}{2} - \frac{1}{2}\sqrt{1-2(1-c)^2} < x < \frac{1}{2} + \frac{1}{2}\sqrt{1-2(1-c)^2}.$$

Proof. The proof is reduced to use of Lemma 2 and check of condition $0.5x(1-x) > \nu$, which is equivalent to what $x \in (a_1, a_2)$, where a_1 and a_2 are the roots of the quadratic equation $0.5x(1-x) = \left(\frac{1-c}{2}\right)^2$.

This completes the proof of Proposition 2.

These results can be interpreted in the following way with respect to the real behaviour of firms. When firms' capacities are small, $c \leqslant \frac{1}{4}$, the firms need to focus on the upper part of the demand curve and put the price $1-2c$. This follows from the symmetric pure strategy Nash equilibrium $(1-2c, 1-2c)$. When firms' capacities are big, $c > \frac{1}{4}$, firms should put the price 0.5. This follows from Propositions 1 and 2.

In Propositions 1 and 2, symmetric equilibria are considered when both firms put the same price x. However, Lemmas 1 and 2 can be used in order to answer the question about existence of nonsymmetric weak active equilibria. Do profiles $(\bar{x}_1, \bar{x}_2)$, $\bar{x}_1 \neq \bar{x}_2$ that are weak active equilibria exist?

Proposition 3. *Assume that* $0 < c \leqslant \frac{1}{4}$. *Then there is no profile* (x_1, x_2) *of actions such that* $x_1 \neq x_2$, $f(x_1, x_2) \geqslant \mu$, $f(x_2, x_1) \geqslant \mu$.

Proof. Suppose that the function f is defined by (3). However, the domain of the function is not the unit square $0 \leqslant x_1 \leqslant 1$, $0 \leqslant x_2 \leqslant 1$ but the unit square with the carved segment $x_1 = x_2$. This means that the values of the function f are ignored for $x_1 = x_2$.

Note that the inequality

$$c < \left(1 - \frac{c}{1-x_2}\right)(1-x_1)$$

holds for $x_1 < 1-2c$, $x_2 < x_1$. Then it follows from (3) that we have the inequality $f(x_1, x_2) < \mu$ for the income of the firm 1 for $x_1 < 1-2c$, $x_2 \neq x_1$. Similarly, we have the inequality $f(x_2, x_1) < \mu$ for the income of the firm 2 for $x_2 < 1-2c$, $x_1 \neq x_2$. So in the future in proving the theorem, we consider only profiles (x_1, x_2) such that $x_1 \geqslant 1-2c$, $x_2 \geqslant 1-2c$.

Note that the inequality

$$c > \left(1 - \frac{c}{1-x_2}\right)(1-x_1) \tag{4}$$

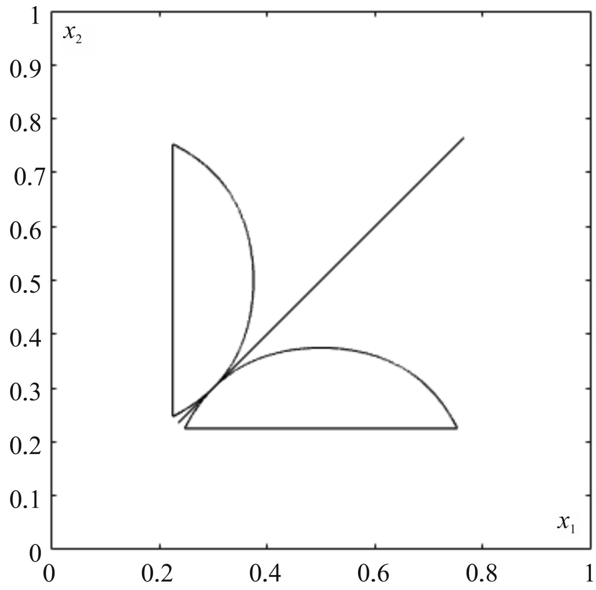

Fig. 1. Weak active equilibria for $c = 0.4$.

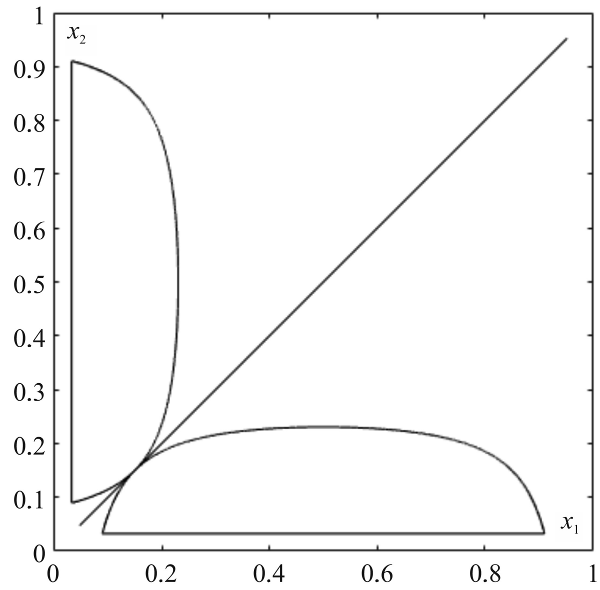

Fig. 2. Weak active equilibria for $c = 0.7$.

holds for $x_1 > 1 - 2c$, $x_2 = 1 - 2c$. Thus, we have $f(x_1, x_2) = 0.5x_1(1 - x_1)$ in this case. Hence, $f(x_1, x_2) < \mu$. Also, inequality (4) holds for $1 - 2c < x_2 \leqslant 1 - c$, $x_1 > x_2$. Thus, $f(x_1, x_2) < 0.5x_1(1 - x_1) < \mu$. Similarly, $f(x_2, x_1) < \mu$ for $1 - 2c \leqslant x_1 \leqslant 1 - c$, $x_2 > x_1$.

Finally, it follows from (3) that $f(x_1, x_2) = 0$ for $x_2 > 1 - c$, $x_1 > x_2$. Similarly, $f(x_2, x_1) = 0$ for $x_1 > 1 - c$, $x_2 > x_1$.

This completes the proof of Proposition 3.

It follows from Lemma 1 and Proposition 3 that there are no nonsymmetric weak active equilibria for $0 < c \leqslant \frac{1}{4}$. Calculations show that nonsymmetric weak active equilibria exist for $c > \frac{1}{4}$. By γ denote the ratio of the number of profiles (x_1, x_2), $x_1 > x_2$ that are weak active equilibria to the total number of profiles (x_1, x_2), $x_1 > x_2$. (The fine grid is used.) Values γ for some c are given in the table.

Values γ for different c

c	0.3	0.4	0.5	0.6	0.7	0.8	0.9
γ	0.0063	0.1113	0.2059	0.2586	0.2684	0.2349	0.1506

It is seen from Figs. 1 and 2 that the set of profiles (x_1, x_2) that are weak active equilibria consists of a part of the diagonal of the unit square (symmetric weak active equilibria) and two lunes that are adjacent to this diagonal (nonsymmetric weak active equilibria).

CONCLUSION

A study of Bertrand–Edgeworth oligopoly is of great practical importance. This is the approach used to analyse competition policy by authorities in a number of countries ([13]). In this paper, the concept of the weak active equilibrium is being used for the first time to study Bertrand–Edgeworth oligopoly. Its use makes it possible to consider only pure strategy equilibria and not consider mixed strategy equilibria like one have to do when using Nash equilibria. All symmetric weak active equilibria for the Beckmann's Bertrand–Edgeworth duopoly model are found. The question of the existence of nonsymmetric weak active equilibria is investigated.

The weak active equilibrium is not unique except for the cases when the weak active equilibrium coincides with the pure strategy Nash equilibrium. In this connection there is an interesting question about the narrowing of the concept of the weak active equilibrium, indication of some properties that allow to select one profile from the set of weak active equilibria. For example, the narrowing of the concept of the weak active equilibrium is the concept of the strong active equilibrium, which is introduced in [17]. But most likely, the latter concept is not suitable for Bertrand–Edgeworth oligopoly.

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